ON THE IMPRESSION OF A RIGID DIE INTO AN ELASTIC SPHERE

(O VDAVLIVANII ZHESTKOGO SHTAMPA V UPRUGUIU SFERU)

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N.Kh. ARUTIUNIAN and B.L. ABRAMIAN
(Yerevan)

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The problem of the impression of a rigid die into an elastic sphere is investigated (Section 1). The solution of this problem reduces to the determination of some coefficients in "coupled" series-equations, containing Legendre polynomials. A method is indicated, which allows the reduction of the solution of the obtained "coupled" series-equations to the solution of an infinite system of linear equations (Section 2).

As an example, the compression of an elastic sphere under the uniform normal loading is investigated (Section 3), with the lower half of the sphere resting in a rigid semispherical recess. Tables and curves are presented for the stresses and displacements.

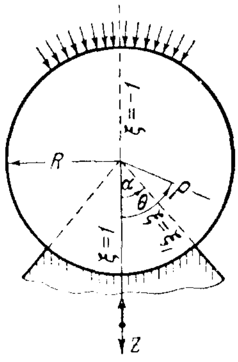

Fig. 1

The problems of the impression of two rigid dies into an elastic sphere and of the axisymmetrical compression of an elastic sphere with a rigid ring girdle, are discussed in other papers by the authors and by Babloian [1 and 2].

1. Let us investigate the axisymmetric problem of the impression of a rigid die into an elastic sphere of radius R (Fig.1).

We shall assume, for simplicity of presentation, that under the die, as well as elsewhere, there are no tangential stresses, and that the normal stresses on the sphere surface are given.

With such formulation, the boundary conditions of the problem, in a spherical coordinate system ρ, θ, φ, will have the form, with $\rho = R$.

$$U_\rho = f^*(\theta) \qquad (0 \leqslant \theta < \alpha)$$
$$\tau_{\rho\theta} = 0 \qquad (0 \leqslant \theta \leqslant \pi), \qquad \sigma_\rho = \psi^*(\theta) \qquad (\alpha < \theta \leqslant \pi) \tag{1.1}$$

Here, U_ρ is the radial component of displacement, $\tau_{\rho\theta}$ and σ_ρ are, respectively, the tangential and normal stresses, $f^*(\theta)$ is a continuous function which determines the shape of the die surface, $\psi^*(\theta)$ is a piecewise continuous function with a limited variation in the indicated interval which prescribes the distribution of normal stresses on the surface of the elastic sphere outside the die, and α is a parameter indicating the size of the die.

The equilibrium equations, in spherical coordinates, with axial symmetry and in the absence of body forces are of the form

$$(\lambda + 2\mu) \sin\theta \frac{\partial\Delta}{\partial\theta} + \mu \frac{\partial}{\partial\rho}(2\rho\omega_\varphi \sin\theta) = 0$$
$$(\lambda + 2\mu)\rho^2 \sin\theta \frac{\partial\Delta}{\partial\rho} - \mu \frac{\partial}{\partial\theta}(2\rho\omega_\varphi \sin\theta) = 0 \tag{1.2}$$

Here λ and μ are Lamé's elastic constants, ω_φ is the rotation component, Δ is the volumetric expansion

$$\omega_\varphi = \frac{1}{2\rho}\left[\frac{\partial}{\partial\rho}(\rho U_\theta) - \frac{\partial U_\rho}{\partial\theta}\right]$$
$$\Delta = \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial\rho}(\rho^2 U_\rho \sin\theta) + \frac{\partial}{\partial\theta}(\rho U_\theta \sin\theta)\right] \tag{1.3}$$

and U_θ is the meridional component of displacement.

Changing from the coordinate θ to the coordinate $\xi = \cos\theta$ and solving Equations (1.2) for the displacements U_ρ and U_θ, we obtain Expressions

$$U_\rho(\rho, \xi) = A_0 \frac{\rho}{R} +$$
$$+ \sum_{k=1}^{\infty}\left\{- kA_k\left(\frac{\rho}{R}\right)^{k-1} - \frac{\lambda k + \mu(k-2)}{\lambda(k+3) + \mu(k+5)}(k+1) C_k\left(\frac{\rho}{k}\right)^{k+1}\right\} P_k(\xi) \tag{1.4}$$

$$U_\theta(\rho, \xi) = \sum_{k=1}^{\infty}\left[A_k\left(\frac{\rho}{R}\right)^{k-1} + C_k\left(\frac{\rho}{R}\right)^{k+1}\right]\sqrt{1-\xi^2}P_k'(\xi) \qquad \left(P_k'(\xi) = \frac{d}{d\xi}P_k(\xi)\right)$$

Here, $P_k(\xi)$ are Legendre polynomials [3], and A_0, A_k and C_k are the integration constants, to be determined form the boundary conditions (1.1).

2. To determine the integration constants, using the relations (1.4) and known equations, expressing the stresses σ_ρ and $\tau_{\rho\theta}$ in terms of the displacement components U_ρ and U_θ, we get

$$\sigma_\rho = \frac{3\lambda + 2\mu}{R}A_0 - \frac{2\mu}{R}\sum_{k=1}^{\infty} P_k(\xi)\left\{k(k-1)A_k\left(\frac{\rho}{R}\right)^{k-2} + \right.$$
$$\left. + \frac{\lambda(k^2 - k - 3) + \mu(k+1)(k-2)}{\lambda(k+3) + \mu(k+5)}(k+1)C_k\left(\frac{\rho}{R}\right)^k\right\} \tag{2.1}$$

$$\tau_{\rho\theta} = \frac{2\mu}{R}\sum_{k=0}^{\infty} P_k'(\xi)\sqrt{1-\xi^2}\left\{(k-1)A_k\left(\frac{\rho}{R}\right)^{k-2} + \right.$$
$$\left. + \frac{\lambda(k+2)k + \mu(k^2 + 2k - 1)}{\lambda(k+3) + \mu(k+5)} C_k\left(\frac{\rho}{R}\right)^k\right\} \tag{2.2}$$

Satisfying the boundary condition (1.1), we obtain the following expression for the coefficients C_k :

$$C_k = -\frac{(k-1)[\lambda(k+3) + \mu(k+5)]}{\lambda k(k+2) + \mu(k^2 + 2k - 1)} A_k \qquad (k = 1, 2, \ldots) \tag{2.3}$$

and the following "coupled" series-equations for the determination of the coefficients A_k :

$$\sum_{k=1}^{\infty} B_k P_k(\xi) = f(\xi) \qquad (1 \geqslant \xi > \xi_1 = \cos\alpha)$$
$$\sum_{k=0}^{\infty} B_k (k-1)\frac{\lambda(2k^2 + 4k + 3) + 2\mu(k^2 + k + 1)}{\lambda(2k+1)k + 2\mu(2k^2 - 1)} P_k(\xi) = \frac{\psi(\xi)}{2\mu} \qquad (\xi_1 > \xi \geqslant -1) \tag{2.4}$$

In addition, we define

$$f^*(\theta) = f(\xi), \qquad \psi^*(\theta) = \frac{\psi(\xi)}{R} \tag{2.5}$$

$$A_0 = B_0, \quad A_k = -\frac{\lambda k(k+2) + \mu(k^2+2k-1)}{\lambda k(2k+1) + 2\mu(2k^2-1)} B_k \qquad (k = 1, 2, 3, \ldots) \tag{2.6}$$

Thus, the solution of the problem under investigation has been reduced to the determination of the unknown coefficients B_k in the "coupled" series-equations (2.4) containing Legendre polynomials.

We shall present the "coupled" series-equations (2.4) in the form

$$\sum_{k=0}^{\infty} B_k P_k(\xi) = f(\xi) \quad (1 \geqslant \xi > \xi_1), \qquad \sum_{k=0}^{\infty} \left(k+\frac{1}{2}\right) B_k P_k(\xi) = F(\xi) \quad (\xi_1 > \xi \geqslant -1) \tag{2.7}$$

where

$$F(\xi) = \frac{\psi(\xi)}{2\mu} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{B_k P_k(\xi)}{\lambda(2k+1)k + 2\mu(2k^2-1)} \times$$

$$\times [3\lambda(k+2) + 2\mu(2k^3+2k^2-2k+1)] \tag{2.8}$$

"Coupled" series-equations of the form (2.7) have been investigated in the work of Babloian [4]. The solution of such series-equations, where $f(\xi)$ and $F(\xi)$ are given functions, is obtained from

$$B_k = \frac{\sqrt{2}}{\pi}\int_0^{\alpha} \cos\left(k+\frac{1}{2}\right)\varphi\, d\varphi \frac{d}{d\varphi}\int_{\cos\varphi}^{1} f(\xi)(\xi - \cos\varphi)^{-1/2} d\xi +$$

$$+ \frac{\sqrt{2}}{\pi}\int_{\alpha}^{\pi} \cos\left(k+\frac{1}{2}\right)\varphi\, d\varphi \int_{-1}^{\cos\varphi} F(\xi)(\cos\varphi - \xi)^{-1/2} d\xi \qquad \begin{pmatrix}\alpha = \cos^{-1}\xi_1 \\ k = 0, 1, 2, \ldots\end{pmatrix} \tag{2.9}$$

Using Equation (2.9) and considering that the unknown coefficient B_k, according to (2.8), enters in the function $F(\xi)$, the solution of the "coupled" series-equations (2.4) for the determination of the integration constants A_k (or the coefficients B_k), after some transformations, is reduced to the solution of an infinite system of linear equations of the following form:

$$A_k = \sum_{p=1}^{\infty} L_{kp} A_p + M_k \qquad (k = 1, 2, \ldots) \tag{2.10}$$

where L_{kp} and M_k are obtained from

$$M_k = -\frac{\sqrt{2}}{\pi}\left(b_k + \frac{a_k b_0}{a_0}\right)\frac{\lambda k(k+2) + \mu(k^2+2k-1)}{\lambda k(2k+1) + 2\mu(2k^2-1)} \qquad (k = 1, 2, \ldots) \tag{2.11}$$

$$L_{kp} = \frac{2}{\pi}\frac{[\lambda k(k+2) + \mu(k^2+2k-1)][3\lambda(p+2) + 2\mu(2p^3+2p^2-2p+1)]}{[\lambda k(2k+1) + 2\mu(2k^2-1)](2p+1)[\lambda p(p+2) + \mu(p^2+2p-1)]} I_{kp}$$

$$(k = 1, 2, \ldots; \; p = 1, 2, \ldots) \tag{2.12}$$

Here

$$\sqrt{2} a_0 = \pi + (3\lambda/\mu + 1)(\pi - \sqrt{1-\xi_1^2} - \cos^{-1}\xi_1) \tag{2.13}$$

$$\sqrt{2} a_k = \left(3\frac{\lambda}{\mu} + 1\right)\left\{\frac{\sin[(k+1)\cos^{-1}\xi_1]}{k+1} + \frac{\sin(k\cos^{-1}\xi_1)}{k}\right\} (k = 1, 2, \ldots) \tag{2.14}$$

$$b_k = \int_0^{\alpha} \cos\left(k+\frac{1}{2}\right)\varphi\, d\varphi \frac{d}{d\varphi}\int_{\cos\varphi}^{1} f(\xi)\,(\xi-\cos\varphi)^{-1/2} d\xi + \tag{2.15}$$

$$+\frac{1}{2\mu}\int_{\alpha}^{\pi}\cos\left(k+\frac{1}{2}\right)\varphi\, d\varphi\int_{-1}^{\cos\varphi}\psi(\xi)\,(\cos\varphi-\xi)^{-1/2}d\xi \quad \left(\begin{matrix}\alpha = \cos^{-1}\xi_1\\ k = 0, 1, 2, \ldots\end{matrix}\right)$$

$$I_{kp} = \frac{a_k}{a_0}\int_{\alpha}^{\pi}\cos\frac{\varphi}{2}\cos\left(p+\frac{1}{2}\right)\varphi\, d\varphi + \int_{\alpha}^{\pi}\cos\left(k+\frac{1}{2}\right)\varphi\cos\left(p+\frac{1}{2}\right)\varphi\, d\varphi \tag{2.16}$$

Solving the infinite system (2.10) and using the obtained values of A_k, the constant A_0 is determined from

$$A_0 = \frac{b_0}{a_0} + \frac{\mu}{a_0(3\lambda+\mu)}\sum_{p=1}^{\infty} A_p \frac{3\lambda(p+2)+2\mu(2p^3+2p^2-2p+1)}{(2p+1)\,[\lambda p(p+2)+\mu(p^2+2p-1)]} a_p \tag{2.17}$$

Equation (2.17) can be obtained directly from (2.9), using $k = 0$ and solving for A_0.

3. As an example, we shall investigate the problem of the compression of an elastic sphere, resting in a rigid semispherical recess, and loaded on its free surface by a uniformly distributed normal load (Fig.2).

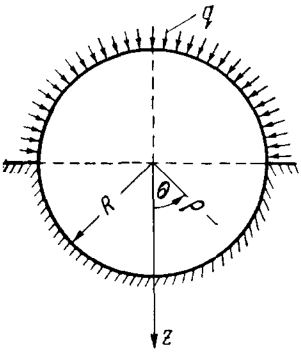

Fig. 2

In the solution of the problem, we assume that the surface of the recess is smooth, i.e. there are no cohesive forces on the contact surface.

The boundary conditions, for this case, are of the form

$$U_\rho = 0 \quad (0 \leqslant \theta \leqslant {}^1/_2\,\pi), \qquad \tau_{\rho\theta} = 0 \quad (0 \leqslant \theta \leqslant \pi)$$

$$\sigma_\rho = -q\,/\,R \quad ({}^1/_2\,\pi < \theta \leqslant \pi) \qquad \text{for } \rho = R \tag{3.1}$$

The integration constants A_k are obtained from the solution (*) of the reduced system (2.10)

$$A_k = \sum_{p=1}^{N} L_{kp}A_p + M_k \qquad (k = 1, 2, \ldots, N) \tag{3.2}$$

The calculations were performed with the values

$$\xi_1 = 0, \quad \frac{\lambda}{\mu} = 2, \quad N = \begin{cases}10\\31\end{cases} \qquad f(\xi) = 0, \quad \psi(\xi) = -q \tag{3.3}$$

We tabulate below some values of $A_k^\circ = A_k\mu\,/\,q$ for a series of k values

$k =$	1	2	3	4	5	6	
$A_k^\circ =$	−0.0872	0.0424	0.0295	−0.0199	−0.0159	0.0130	($N = 10$)
	−0.0883	0.0431	0.0307	−0.0199	−0.0166	0.0128	($N = 31$)

$k =$	7	8	9	10	11	12	
$A_k^\circ =$	0.0106	−0.0097	−0.0078	0.0082	—	—	($N = 10$)
	0.0112	−0.0094	−0.0083	0.0074	0.0066	−0.0061	($N = 31$)

*) The calculations were performed at the Computer Center of the ArmSSR Academy of Science and the Yerevan state University by the center coworker A.Bardanian and processed by the coworker of the Institute of Mathematics and Mechanics A.A.Babloian. The authors regard it as their pleasant duty to record their thanks.

Table continued

$k = \;\; 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18$

$A_k{}^\circ = -0.0055 \quad 0.0052 \quad 0.0047 \quad -0.0046 \quad -0.0041 \quad 0.0041 \quad (N = 31)$

$k = \;\; 19 \quad 20 \quad 21 \quad 22 \quad 23 \quad 24$

$A_k{}^\circ = 0.0036 \quad -0.0037 \quad -0.0032 \quad 0.0033 \quad 0.0029 \quad -0.0031 \quad (N = 31)$

$k = \;\; 25 \quad 26 \quad 27 \quad 28 \quad 29 \quad 30 \quad 31$

$A_k{}^\circ = -0.0026 \quad 0.0029 \quad 0.0023 \quad -0.0027 \quad -0.0021 \quad 0.0026 \quad 0.0018 \quad (N = 31)$

Using these values, and Equation (2.17), we get

$$A_0 = -0.0976\, q / \mu \quad \text{for } N = 10, \qquad A_0 = -0.0998\, q / \mu \quad \text{for } N = 31 \qquad (3.4)$$

Calculating the C_k coefficients from Equation (2.3), the stresses σ_ρ, $\tau_{\rho\theta}$ and the displacements at any point in the sphere can be determined from Equations (2.1), (2.2) and (1.4).

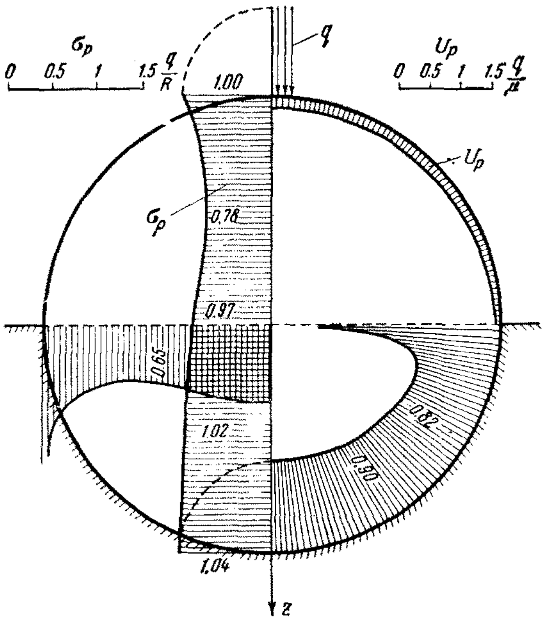

Fig. 3

We present the values of the stress $\sigma_\rho{}^\circ = \sigma_\rho R / 2q$, calculated at some points in the sphere, and also the values of the displacenents $U_\rho{}^\circ = U_\rho \mu / q$ and $U_\theta{}^\circ = U_\theta \mu / q$, calculated at some points on the sphere surface and the equatorial plane

	$(R, 1)$	$(^1/_2 R, 1)$	$(0,1)$	$(^1/_2 R, -1)$	$(R, {}^1/_2\sqrt{3})$	$(R, {}^1/_2)$	
$\sigma_\rho =$	-0.503	-0.497	-0.475	-0.385	-0.484	-0.433	$(N = 10)$
	-0.517	-0.510	-0.486	-0.392	-0.451	-0.419	$(N = 31)$

	$(^1/_2 R, {}^1/_2\sqrt{3})$	$(^1/_2 R - {}^1/_2)$	$(^1/_2 R, 0)$	$(^1/_2 R, -{}^1/_2)$	$(^1/_2 R, -{}^1/_2\sqrt{3})$	
$\sigma_\rho =$	-0.469	-0.366	-0.318	-0.436	-0.408	$(N = 10)$
	-0.480	-0.373	-0.326	-0.446	-0.417	$(N = 31)$

Table continued

$$\begin{array}{llllll} & (R,\ -1) & (R, -\frac{1}{2}\sqrt{3}) & (R,\ -\frac{1}{2}) & (R,\ 0) & \\ U_\rho^\circ = \Big\{ & -0.210 & -0.191 & -0.173 & - & (N=10) \\ & -0.197 & -0.192 & -0.171 & -0.018 & (N=31) \end{array}$$

$$\begin{array}{lllll} & (R,\ 0) & (\frac{1}{2}R,\ 0) & (0,0) & \\ U_\theta^\circ = \Big\{ & -0.090 & -0.098 & -0.087 & (N=10) \\ & -0.088 & -0.099 & -0.088 & (N=31) \end{array}$$

As a pictorial representation of the distribution of normal stresses, Fig.3 shows the curves of the normal stresses σ_ρ .

We should note that the investigation of the question of the regularity of the infinite system of linear equations (2.10), or the reduction of this system to a regular system [5], presents mathematical difficulties.

To obtain an approximate solution, an abridged system of equations (3.2) was used. This system was solved with $N = 10$ and $N = 31$, where N is the number of equations in the abridged system. The calculations show that the values of the stresses and displacements, presented above for these two cases, differ by a negligible amount.

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